Notes on the Clauser-Horne-Shimony-Holt (CHSH) Inequality

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> *But trivialities like this, you will exclaim, are of no interest in consumer research!* (Bell, 2004 [1981]).

For a system in the singlet state ($|\Psi^-\rangle$), the expectation value for joint experiments on its subsystems is given by the following expression:

$$\langle \sigma_m \otimes \sigma_n \rangle = -\hat{m} \cdot \hat{n} = -\cos\theta. \tag{1}$$

Here σ_m, σ_n represent spin-m and spin-n experiments on the first (Alice's) and second (Bob's) subsystem, respectively, with $\hat{m}, \hat{n}$ the unit vectors representing the orientations of the two experimental devices, and θ the difference in these orientations. Note, in particular, that when $\theta = 0$, $\langle \sigma_m \otimes \sigma_n \rangle = -1$ (i.e., experimental results for the two subsystems are perfectly anti-correlated), when $\theta = \pi$, $\langle \sigma_m \otimes \sigma_n \rangle = 1$ (i.e., experimental results for the two subsystems are perfectly correlated), and when $\theta = \pi/2$, $\langle \sigma_m \otimes \sigma_n \rangle = 0$ (i.e., experimental results for the two subsystems are not correlated at all).

Consider the following attempt (Bell, 2004 [1964]) to reproduce the quantum mechanical predictions for this state by means of a hidden variables theory. Let the hidden variables of the theory assign, at state preparation, to each subsystem of a bipartite quantum system, a unit vector $\hat{\lambda}$ (the same value for $\hat{\lambda}$ is assigned to each subsystem) which determines the outcomes of subsequent experiments on the system as follows. Let the functions

$A_\lambda(\hat{m}), B_\lambda(\hat{n})$ represent, respectively, the outcome of a spin-m and a spin-n experiment on Alice's and Bob's subsystem. Define these as:

$$A_\lambda(\hat{m}) = \text{sign}(\hat{m} \cdot \hat{\lambda}),$$
$$B_\lambda(\hat{n}) = -\text{sign}(\hat{n} \cdot \hat{\lambda}),$$

where sign(x) is a function which returns the sign (+, -) of its argument.

The reader can verify that the probability that both $A_\lambda(\hat{m})$ and $B_\lambda(\hat{n})$ yield the same value, and the probability that they yield values that are different (assuming a uniform probability distribution over $\hat{\lambda}$), are respectively:

$$\Pr(+,+) = \Pr(-,-) = \theta/2\pi,$$
$$\Pr(+,-) = \Pr(-,+) = \frac{1}{2}\left(1 - \frac{\theta}{\pi}\right),$$

with θ the (positive) angle between $\hat{m}$ and $\hat{n}$. This yields, for the expectation value of experiments on the combined state:

$$\langle \sigma_m \otimes \sigma_n \rangle = \frac{2\theta}{\pi} - 1.$$

When θ is a multiple of $\pi/2$, this expression yields predictions identical to the quantum mechanical ones: perfect anti-correlation for $\theta \in \{0, 2\pi, ...\}$, no correlation for $\theta \in \{\pi/2, 3\pi/2, ...\}$, and perfect correlation for $\theta \in \{\pi, 3\pi, ...\}$. However, for all other values of θ there are divergences from the quantum mechanical predictions.

It turns out that this is not a special characteristic of the simple hidden variables theory considered above. *No* hidden variables theory is able to reproduce the predictions of quantum mechanics if it makes the very reasonable assumption that the probabilities of local experiments on Alice's subsystem (and likewise Bob's) are completely determined by Alice's local experimental setup together with a hidden variable taken on by the subsystem at the time the joint state is prepared.[1] Consider the following[2] expression relating different spin experiments on Alice's and Bob's respective subsystems for arbitrary directions $\hat{m}, \hat{m}', \hat{n}, \hat{n}'$:

$$|\langle \sigma_m \otimes \sigma_n \rangle + \langle \sigma_m \otimes \sigma_{n'} \rangle| + |\langle \sigma_{m'} \otimes \sigma_n \rangle - \langle \sigma_{m'} \otimes \sigma_{n'} \rangle|. \quad (2)$$

[1]It is also (reasonably) assumed that Alice's and Bob's local measurements occur at spacelike and not timelike separation. For further discussion, see: Kent (2005).

[2]In this exposition of the CHSH inequality I have followed Myrvold (2008).

As before, let $A_\lambda(\hat{m}) \in \{\pm 1\}, B_\lambda(\hat{n}) \in \{\pm 1\}$ represent the results, given a specification of some hidden variable λ, of spin experiments on Alice's and Bob's subsystems. We make no assumptions about the nature of the 'common cause' λ this time—it may take any form. What we do assume is that, as I mentioned above, the outcomes of Alice's experiments depend only on her local setup and on the value of λ; i.e., we do not assume any further dependencies between Alice's and Bob's local experimental configurations. This 'factorisability' allows us to substitute $\langle A_\lambda(\hat{m}) \cdot B_\lambda(\hat{n}) \rangle$ for $\langle \sigma_m \otimes \sigma_n \rangle$, thus yielding:

$$\begin{aligned}
&\big|\big\langle A_\lambda(\hat{m}) B_\lambda(\hat{n}) \big\rangle + \big\langle A_\lambda(\hat{m}) B_\lambda(\hat{n}') \big\rangle\big| + \big|\big\langle A_\lambda(\hat{m}') B_\lambda(\hat{n}) \big\rangle - \big\langle A_\lambda(\hat{m}') B_\lambda(\hat{n}') \big\rangle\big| \\
&= \big|\big\langle A_\lambda(\hat{m})\big(B_\lambda(\hat{n}) + B_\lambda(\hat{n}')\big) \big\rangle\big| + \big|\big\langle A_\lambda(\hat{m}')\big(B_\lambda(\hat{n}) - B_\lambda(\hat{n}')\big) \big\rangle\big| \\
&\leq \big\langle \big| A_\lambda(\hat{m})\big(B_\lambda(\hat{n}) + B_\lambda(\hat{n}')\big) \big| \big\rangle + \big\langle \big| A_\lambda(\hat{m}')\big(B_\lambda(\hat{n}) - B_\lambda(\hat{n}')\big) \big| \big\rangle,
\end{aligned}$$

which, since $|A_\lambda(\cdot)| = 1$, is

$$\begin{aligned}
&\leq \big\langle \big| B_\lambda(\hat{n}) + B_\lambda(\hat{n}') \big| \big\rangle + \big\langle \big| B_\lambda(\hat{n}) - B_\lambda(\hat{n}') \big| \big\rangle \\
&\leq 2,
\end{aligned}$$

where the last inequality follows from the fact that $B_\lambda(\cdot)$ can also only take on values of ± 1. This expression, a variant of the 'Bell inequality' (2004 [1964]), is known as the *Clauser-Horne-Shimony-Holt* (CHSH) inequality (cf., Clauser et al., 1969; Bell, 2004 [1981]).

Quantum mechanics violates the CHSH inequality for some experimental configurations. For example, let the system be in the singlet state; i.e., such that its statistics satisfy (1); and let the unit vectors $\hat{m}, \hat{m}', \hat{n}, \hat{n}'$ (taken to lie in the same plane) have the orientations $0, \pi/2, \pi/4, -\pi/4$ respectively. The differences, θ, between the different orientations (i.e., $\hat{m} - \hat{n}, \hat{m} - \hat{n}', \hat{m}' - \hat{n}$, and $\hat{m}' - \hat{n}'$) will all be in multiples of $\pi/4$ and we will have:

$$\begin{aligned}
&\langle \sigma_m \otimes \sigma_n \rangle = \langle \sigma_m \otimes \sigma_{n'} \rangle = \langle \sigma_{m'} \otimes \sigma_n \rangle = \sqrt{2}/2, \\
&\langle \sigma_{m'} \otimes \sigma_{n'} \rangle = -\sqrt{2}/2, \\
&|\langle \sigma_m \otimes \sigma_n \rangle + \langle \sigma_m \otimes \sigma_{n'} \rangle| + |\langle \sigma_{m'} \otimes \sigma_n \rangle - \langle \sigma_{m'} \otimes \sigma_{n'} \rangle| = 2\sqrt{2} \nleq 2.
\end{aligned}$$

The predictions of quantum mechanics for arbitrary orientations $\hat{m}, \hat{m}', \hat{n}, \hat{n}'$ cannot, therefore, be reproduced by a hidden variables theory in which all correlations between subsystems are due to a common parameter endowed to them at state preparation. *They can*, however, be reproduced by such a hidden variables theory for certain special cases. In particular, the inequality is *satisfied* (as the reader can verify) when $\hat{m}$ and $\hat{n}$, $\hat{m}$ and $\hat{n}'$, $\hat{m}'$ and $\hat{n}$, and $\hat{m}'$ and $\hat{n}'$ are all oriented at angles with respect to one another that are given in multiples of $\pi/2$.

References

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